

# Topological aspects of Cardy-Rabinovici model

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Based on 2009.10183 with Masazumi Honda (YITP)

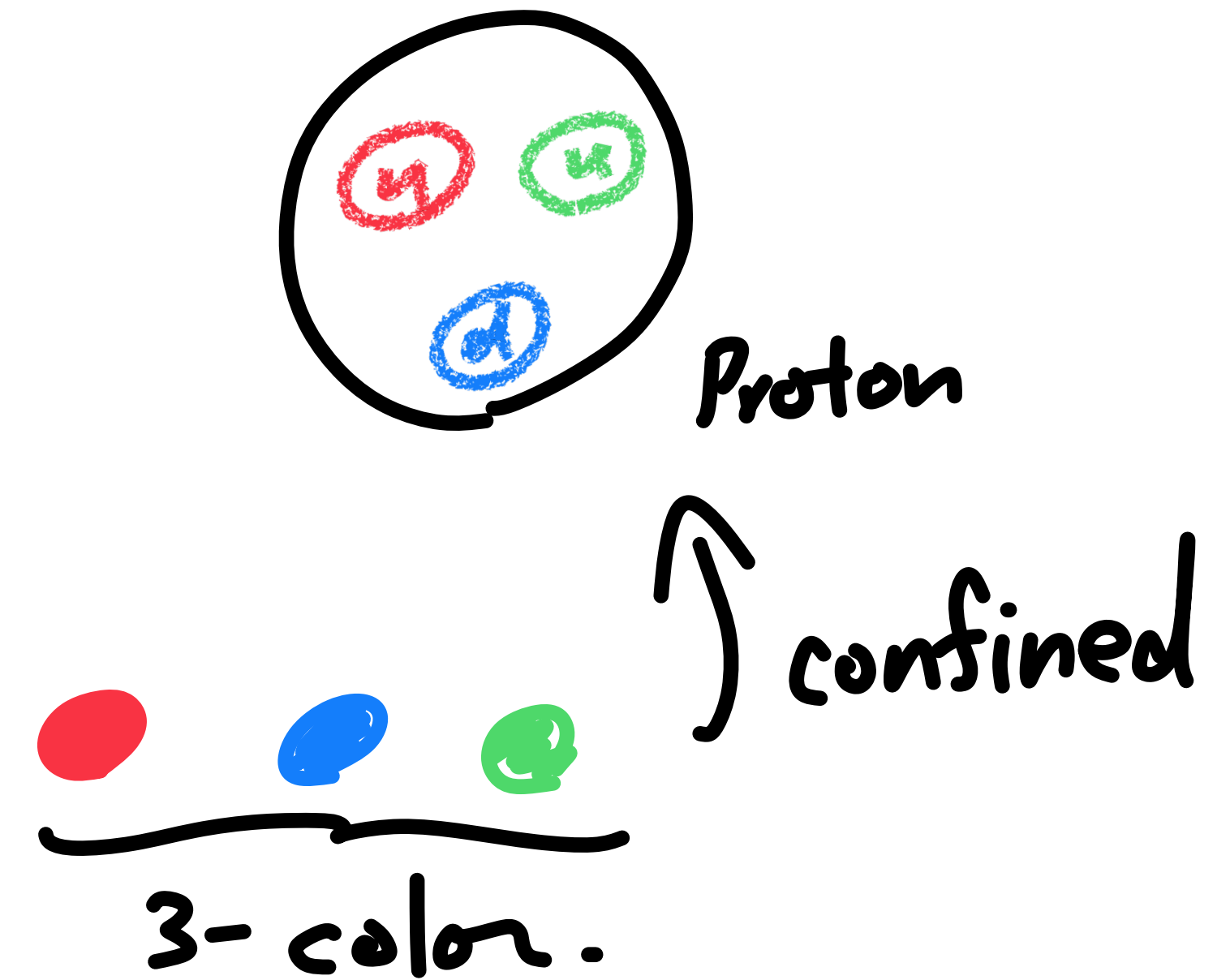
Motivation

Yang-Mills theory &  $\theta$ -angles

# Quark confinement

Strong interaction =  $SU(3)$  non-Abelian gauge theory

Quark (fermion) is in its fund. rep.

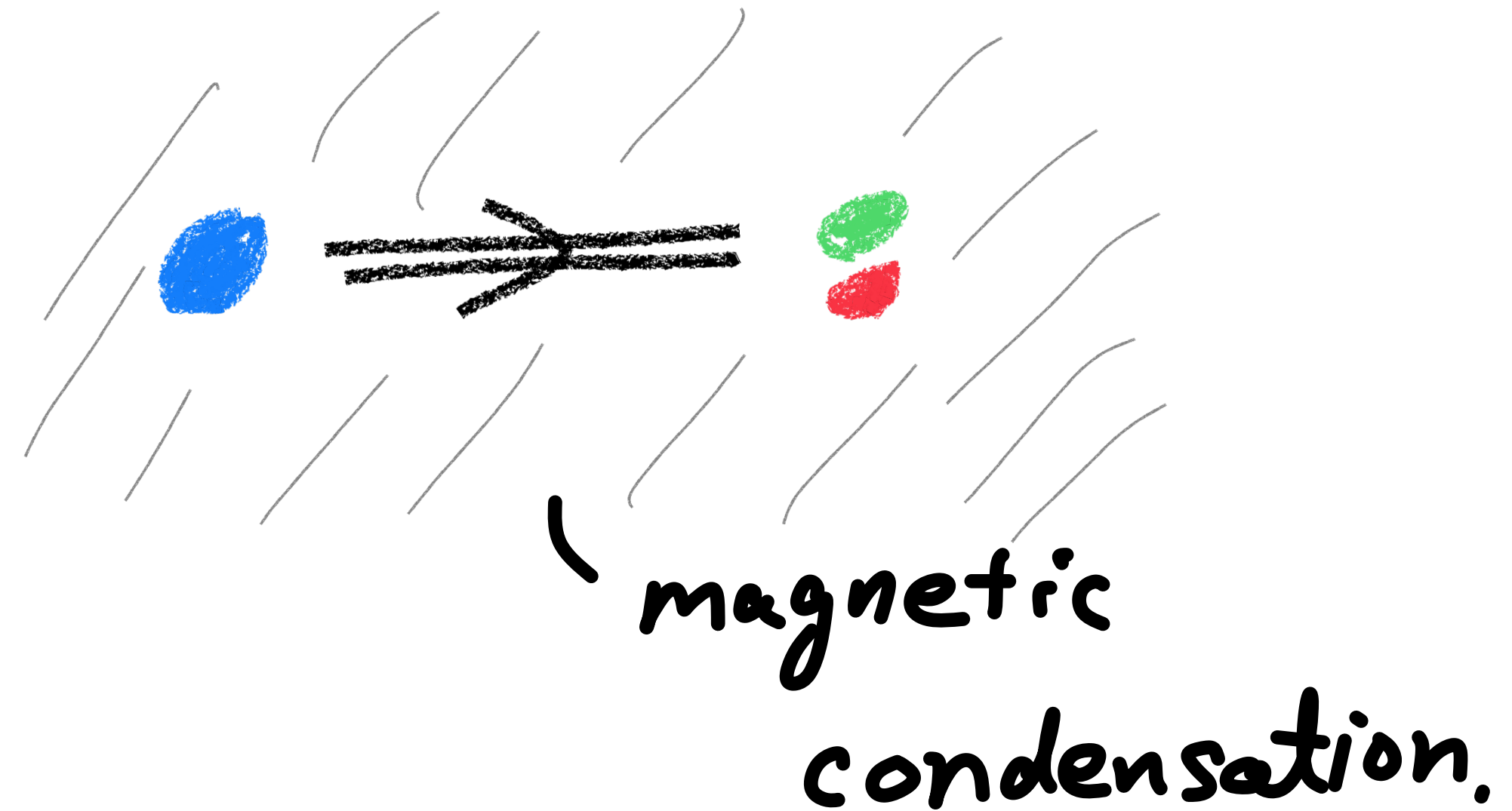


Why such colored particles does not produce asymptotic states?

# Monopole condensation.

Nambu, Mandelstam, 't Hooft ('70s)

Confinement is caused by  
dual superconductivity.



Bunch of monopoles condense into the vacuum.

Such vacuum is also interesting from topological aspects.

Nontrivial topology

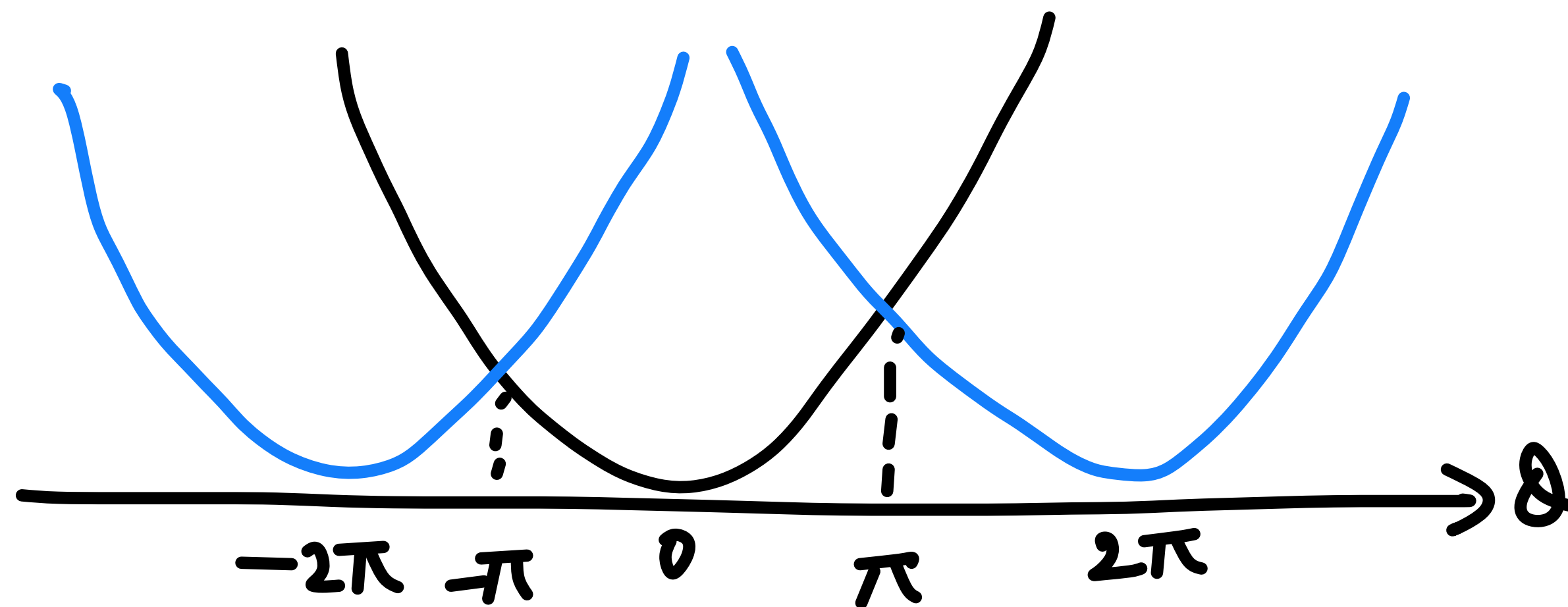
$SU(N)$  Yang-Mills theory: Theory of gluons

$$\mathcal{L} = \frac{1}{g^2} \text{tr}(F_{\mu\nu} F_{\mu\nu}) + i\theta \frac{1}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu} F_{\rho\sigma})$$

instanton #

= winding #  $\in \mathbb{Z}$

Vacuum energy has a nontrivial  $\theta$  dependence:



(Witten '80, '98,  
Di Vecchia, Veneziano '80)

Large- $N$  limit & Spontaneous CP breaking

Large- $N$ :  $N \rightarrow \infty$  with  $\lambda = g^2 N$  fixed.

$$e^{-\beta \cdot \text{Vol} \cdot E(\theta)} = \int \mathcal{D}a \ e^{-N \left( \frac{1}{2\lambda} \int F \wedge *F + i \frac{\theta/N}{8\pi^2} \int F \wedge F \right)}$$

*d.o.f.*

$$\Rightarrow E(\theta) = N^2 f\left(\frac{\theta}{N}\right)$$

$$= N^2 f(0) + \frac{\chi_{\text{top}}}{2} \theta^2 + \mathcal{O}\left(\frac{1}{N^2}\right)$$

To achieve  $\theta \sim \theta + 2\pi$ ,

$$E(\theta) = \min_n \left( \frac{\chi_{\text{top}}}{2} (\theta - 2\pi n)^2 \right).$$

$\Rightarrow n=0$  &  $1$  have the same energy at  $\theta = \pi$ . *CP*

# Anomaly matching

Recently, a new anomaly has been found for YM :

$B$ :  $\mathbb{Z}_N$  2-form gauge field (= 't Hooft flux)

$$\mathbb{Z}_{\theta=\pi}[B] \xrightarrow{\text{CP}} e^{-i\frac{N}{4\pi} \int B \wedge B} \mathbb{Z}_{\theta=\pi}[B]$$

(Gaiotto, Kapustin, Komargodski, Seiberg '17, Tanizaki, Kikuchi '17, ...)

→ ~~CP~~ is indeed natural to satisfy the anomaly matching.

Q.) what are other candidates to achieve anomaly matching?



Model

Cardy - Rabinovici model



# Lattice $U(1)$ gauge theory w/ confinement - deconfinement

(Banks, Myerson, Kogut '77)  
(Savit '77 ...)

$$\begin{cases} a_\mu : \mathbb{R} - \text{valued link variable} \\ S_{\mu\nu} : \mathbb{Z} - \text{valued plaquette variable} \end{cases}$$

$$\Rightarrow f = da - 2\pi S : \text{field strength}$$

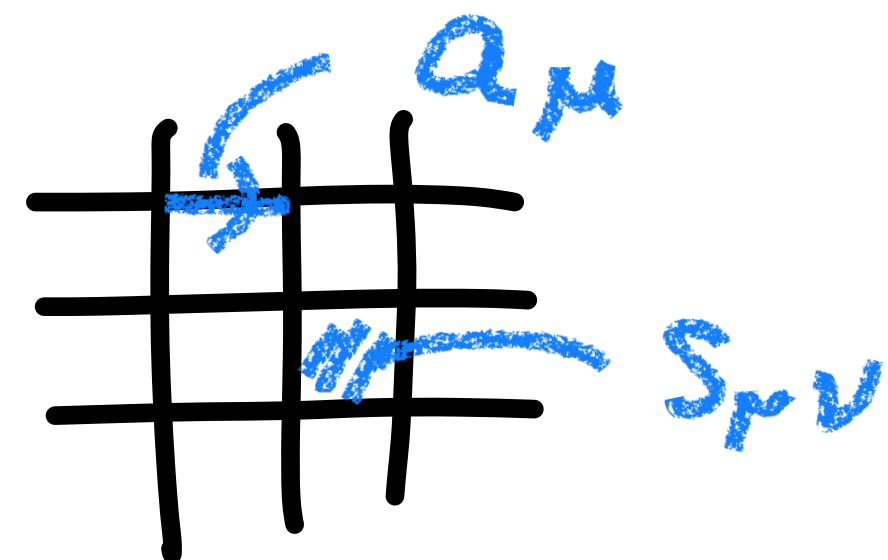
$$\mathcal{L} = \frac{1}{2g^2} \int f \wedge *f + iN \int n_\mu a_\mu$$

↑ world-line of electric charges

\* Theory inherits magnetic particles

$$m = \frac{1}{2\pi} * df = * dS$$

Cardy - Rabinovici ('82):  $\theta$ -angle via Witten effect " $n_\mu \Rightarrow n_\mu + \frac{\theta}{2\pi} m_\mu$ "



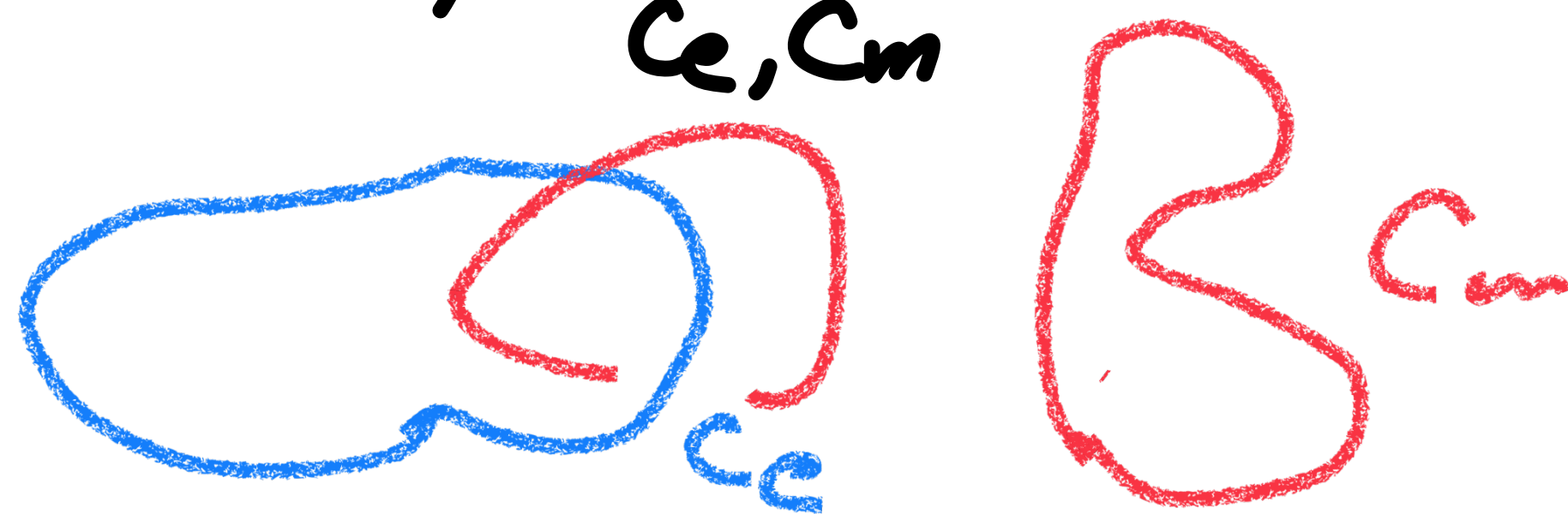
# Cardy - Rabinovici model

In a (formal) continuum description,

$$Z = \int \mathcal{D}a \exp \int \left( -\frac{1}{2g^2} |f|^2 + i \frac{N\theta}{8\pi^2} \int f \wedge f \right) \times \sum_{\text{world-lines } C_e, C_m} W^N(C_e) \cdot H(C_m)$$

Wilson line

't Hooft line



Witten effect

$$(Nn, m) \xrightarrow{\theta} \left( N \left( n + \frac{\theta}{2\pi} m \right), m \right).$$

# Vacuum energy of CR model

If a particle of  $(n, m)$  condenses,

$$F_{(n,m)} \sim g^2 \left( N \left( n + \frac{\theta}{2\pi} m \right) \right)^2 + \left( \frac{2\pi}{g} \right)^2 m^2$$

$$= \frac{N}{\text{Im}(\tau)} |n + m\tau|^2 \quad \left( \tau = \frac{\theta}{2\pi} + i \frac{2\pi}{Ng^2} \right)$$

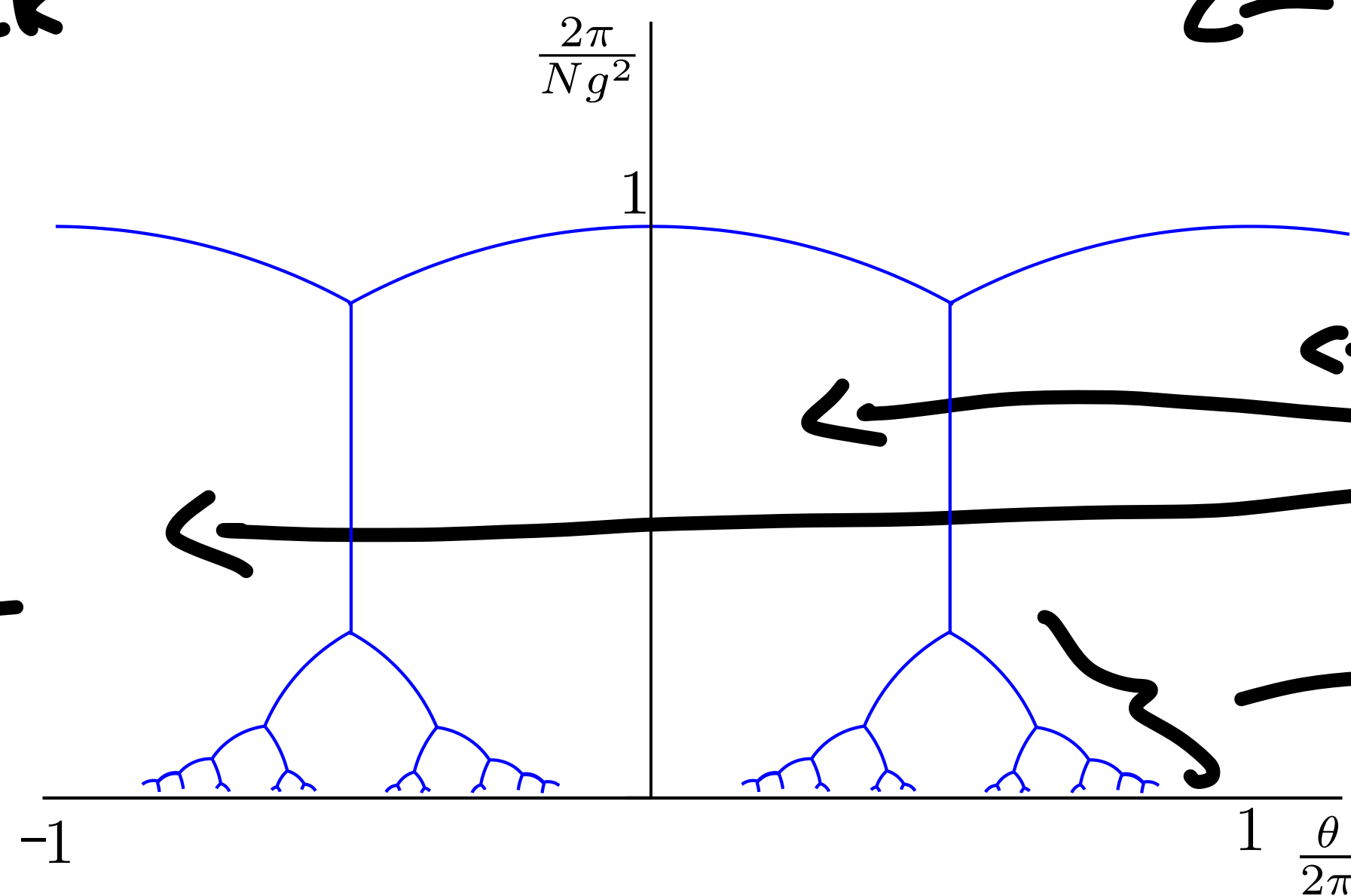
Predicted phases

weak  
↑  
↓  
strong

← Higgs

← Confinement

Oblique confinement



Anomaly Topological aspects of CR model

't Hooft anomaly of CR model.

This model has the same anomaly with pure YM [GKKS'17]

$B$ : Gauge field for  $\mathbb{Z}_N$  1-form symmetry

$$Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B} Z_{\theta}[B]$$

This anomaly can be used to constrain the possible phase diagrams.

$\theta \rightarrow \theta + 2\pi$  : ( $\theta$ -term is  $i \frac{N\theta}{8\pi^2} \int f \wedge f$ ).

$$\Delta S = 2\pi i \frac{N}{8\pi^2} \int (f - B) \wedge (f - B)$$

$$= 2\pi i \left( \underbrace{\frac{N}{8\pi^2} \int f \wedge f}_{\in N\mathbb{Z}} - \underbrace{\frac{N}{4\pi^2} \int f \wedge B}_{\in \mathbb{Z}} + \frac{N}{8\pi^2} \int B \wedge B \right)$$

$$= i \frac{N}{4\pi} \int B \wedge B.$$

(Under this transformation,  $H \rightarrow H W^{-N}$  by Witten eff.,)  
so  $\{n_\mu, m_\mu\} \rightarrow \{n_\mu - m_\mu, m_\mu\}.$

# Anomaly matching in confined phase

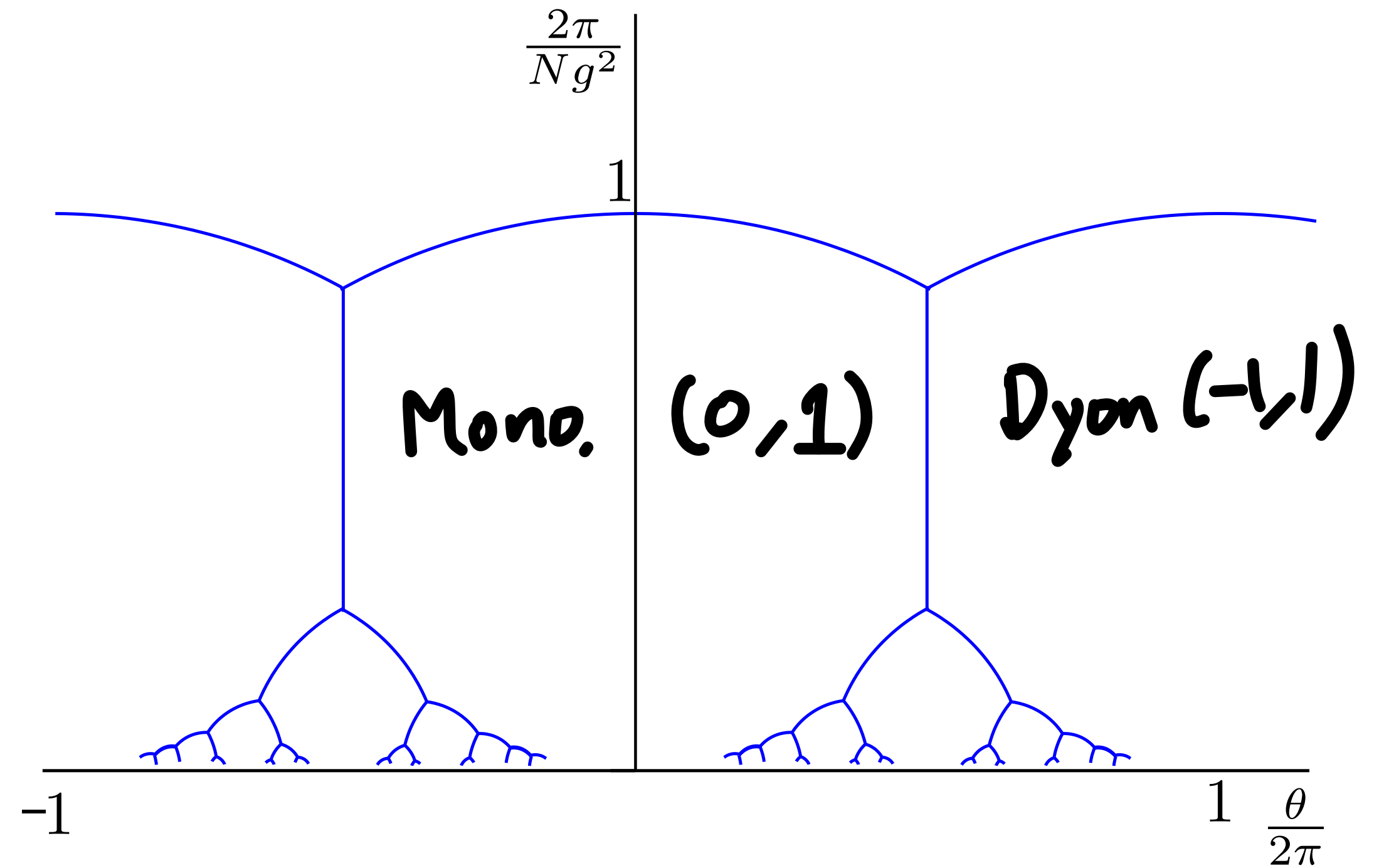
$\theta \approx 0$  : Monopole condensation

$$Z_0[B] \sim 1$$

$\theta \approx 2\pi$  : Dyon condensation

$$Z_{2\pi}[B] \sim e^{i\frac{N}{4\pi} \int B \wedge B}$$

These two vacua are distinct as SPT phases  
with  $\mathbb{Z}_N$  1-form sym.





Exotic condensation : Oblique confinement.

Around  $\theta = \pi$ , exotic condensation is more preferred:

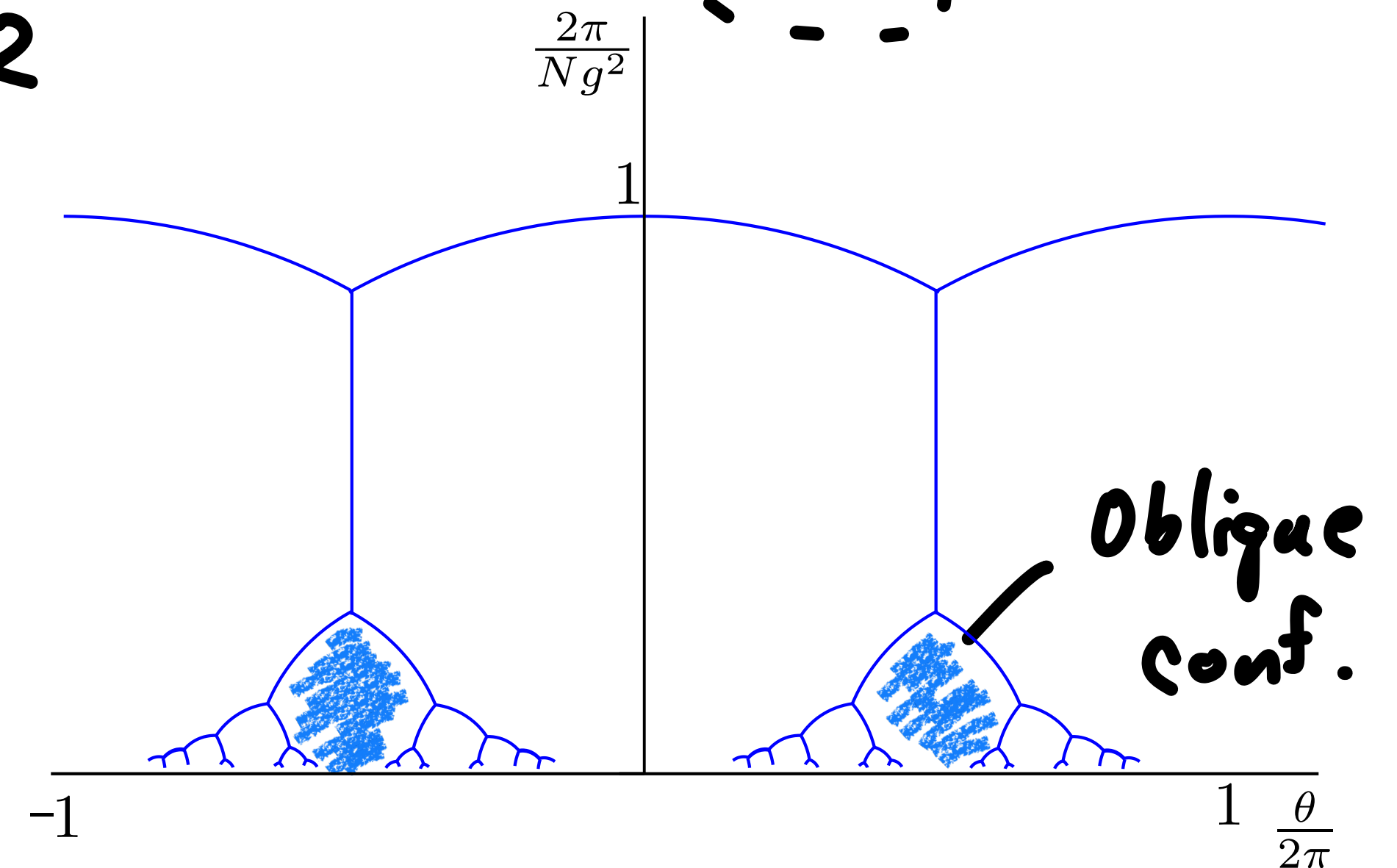
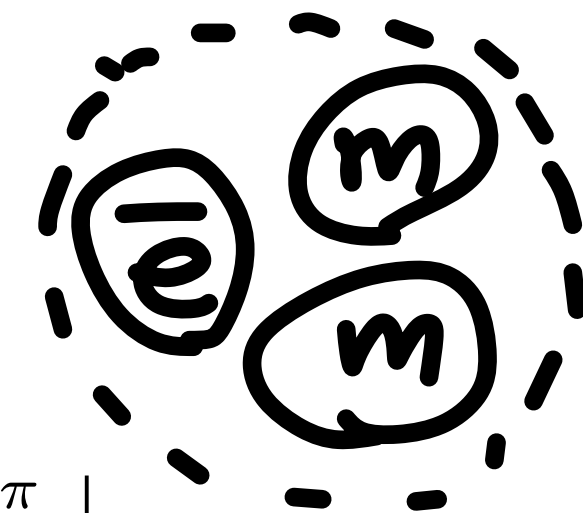
$$F_{\text{mono}} = F_{(0,1)} \sim N^2 g^2 \left( \frac{\theta}{2\pi} \right)^2$$

$\approx \frac{1}{2}$  at  $\theta = 0$

Consider a composite particle  $(-1, 2)$

$$F_{\text{oblique}} = F_{(-1,2)} \sim N^2 g^2 \left( \underbrace{-1 + 2 \cdot \frac{\theta}{2\pi}}_{=0} \right)^2$$

at  $\theta = \pi$



Low-energy property of oblique confinement

$N$ : even int.       $\mathbb{Z}_N^{(1)} \xrightarrow{\text{SSB}} \mathbb{Z}_{N/2}^{(1)}$ .

Only  $W^{\frac{N}{2}}(C)$  is deconfined.  $\rightarrow \mathbb{Z}_2$  topological order.

$N$ : odd int.      All non-trivial Wilson loops are confined.

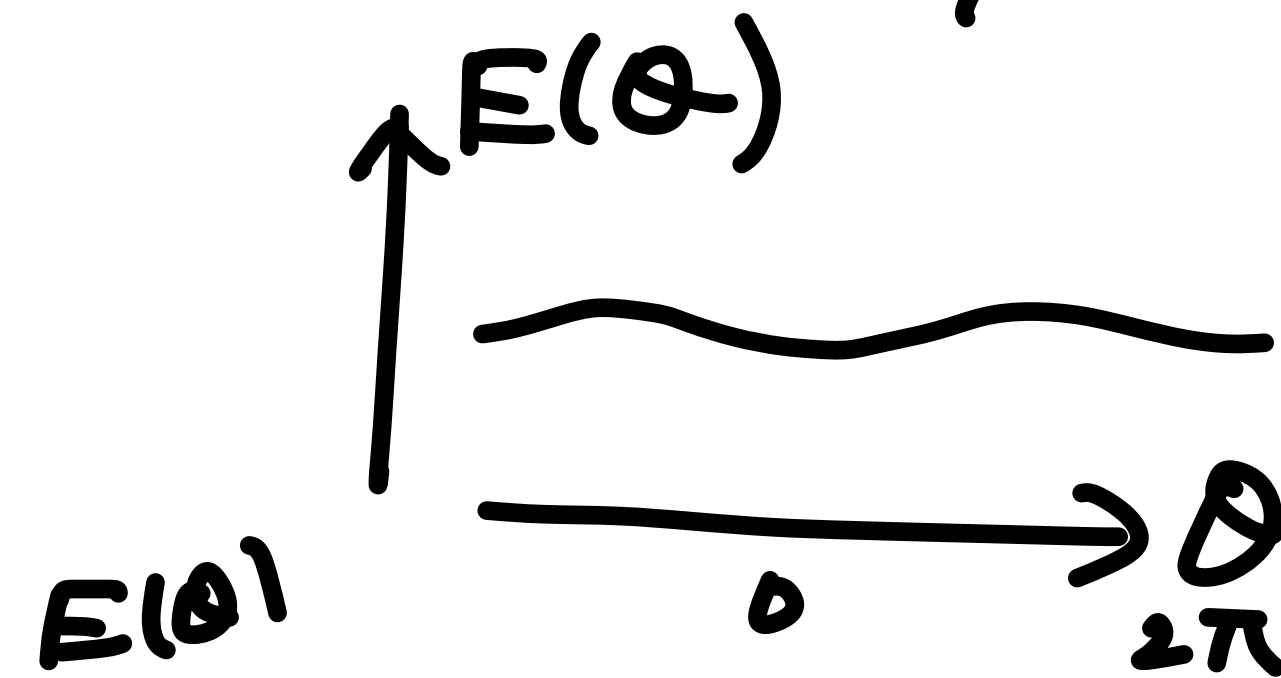
It's an SPT with  $\mathbb{Z}_N^{(1)}$ ,

$$Z[B] \sim e^{i \frac{N-1}{2} \cdot \frac{N}{4\pi} \int B^2}.$$

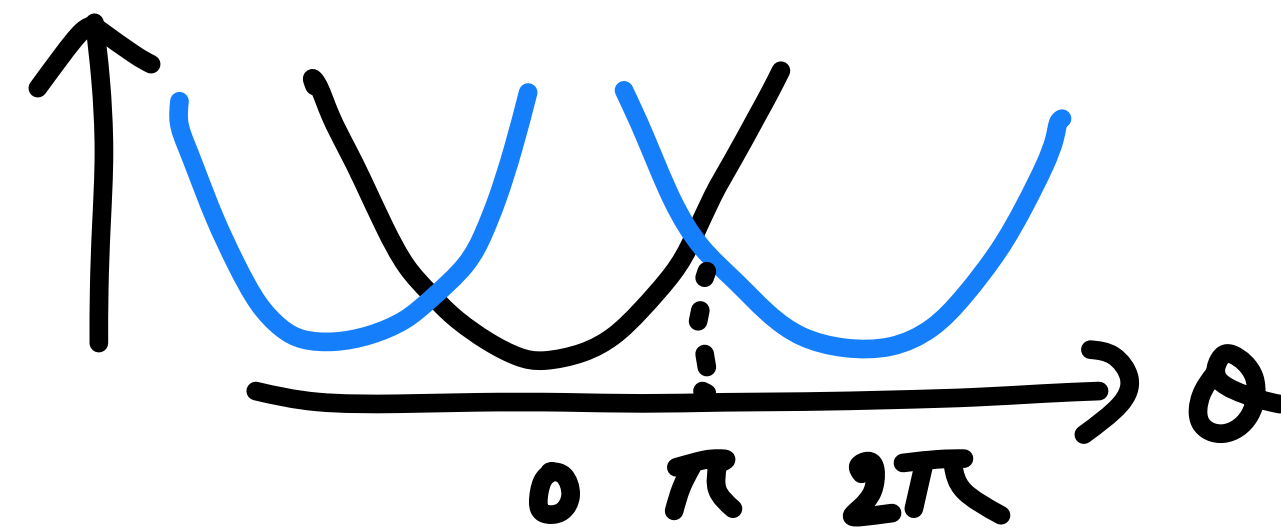
(cf. 't Hooft '81 for  $SU(2)$  gauge theory)

To match  $Z_N^{(1)} \times CP$  anomaly, we can have the following options, which are concretely realized in CR model:

• Higgs  $Z_N^{(1)} \rightarrow 1$   
(or Coulomb)

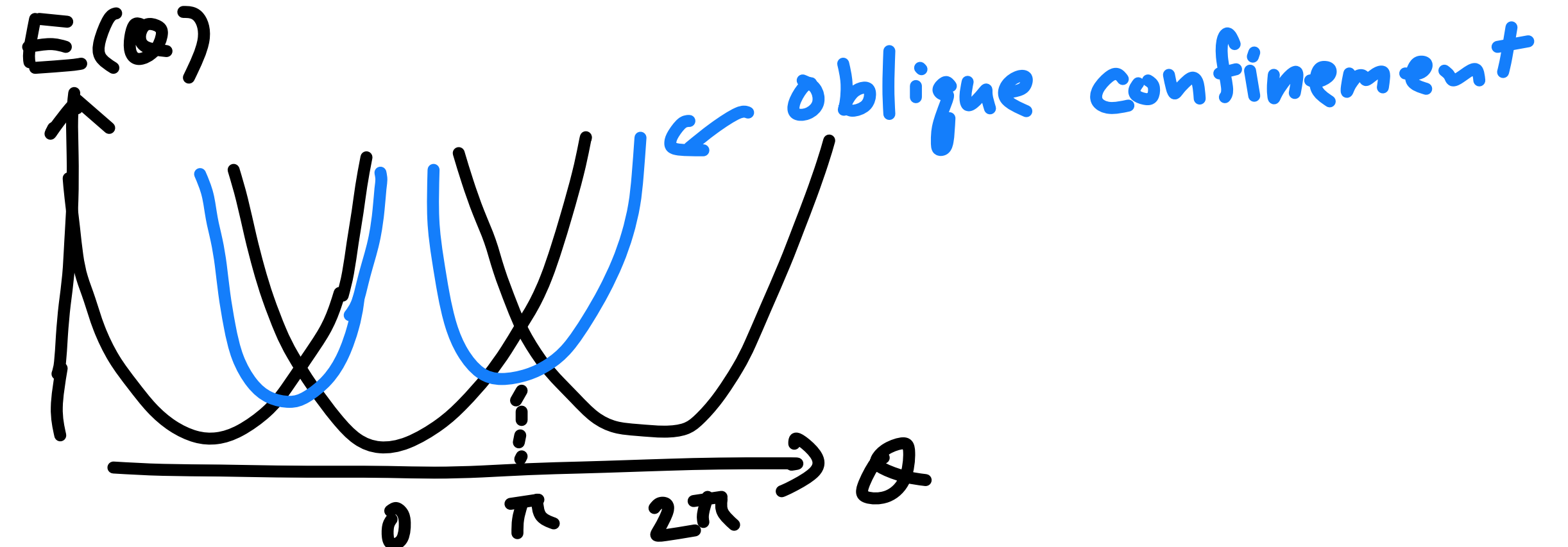


• Confinement ~~CP~~



• Oblique confinement  
 $Z_N^{(1)} \rightarrow Z_{N/2}^{(1)}$  ( $N$ : even)

$Z_N^{(1)}, CP$  ( $N$ : odd)



Self-duality  $SL(2, \mathbb{Z})$ -duality and gravitational anomaly

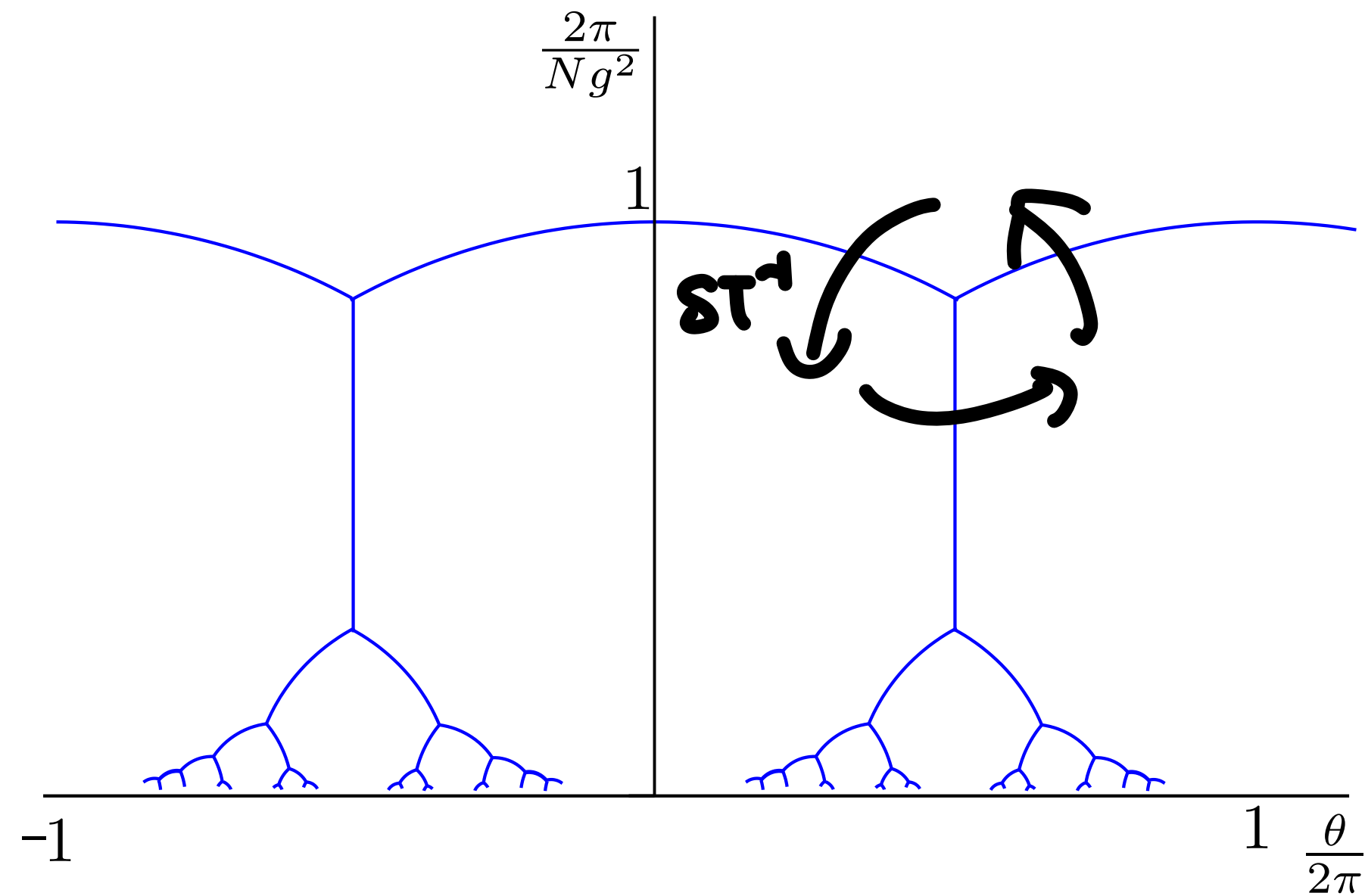
# $SL(2, \mathbb{Z})$ electromag. duality of CR model

Free energy  $F(n, m) = \frac{1}{\text{Im}(\tau)} |n + m\tau|^2$  is inv. under

$$SL(2, \mathbb{Z}) \begin{cases} S: \tau \rightarrow -\frac{1}{\tau}, & (n, m) \mapsto (-m, n) \\ T: \tau \rightarrow \tau + 1, & (n, m) \mapsto (n - m, m) \end{cases}$$

Especially,  $ST^{-1}$  has a fixed point

$$\tau = \tau_* = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$



This  $SL(2, \mathbb{Z})$  is **NOT** a "self"-duality  
as 1-form sym. is not preserved. (Similar is true for  $\mathcal{N}=4$  SYM,  
cf. Gaiotto, Kapustin, Seiberg, Willet '19)

When  $N=M^2$ , we can obtain **self-dual** CR model  
by gauging  $\mathbb{Z}_M^{(1)} \subset \mathbb{Z}_{M^2}^{(1)}$ .

$$Z_{\text{gauged}} = \int \mathcal{D}a \exp \left( - \int \left( \frac{1}{2M^2 g^2} da \wedge * da + i \frac{\theta}{8\pi^2} da \wedge da \right) \right) \\ \times \sum_{\{C_e, C_m\}} \tilde{W}^M(C_e) \tilde{H}^M(C_m).$$

$\leadsto$  Can we use this **self-duality** to constrain the phase str.?



# Mixed $SL(2, \mathbb{Z})$ - gravity anomaly.

It's been known that Maxwell partition func. is a modular form,  
instead of being modular inv. (Witten '95, Verlinde '95)  
(Seiberg, Tachikawa, Yonekura '18).

$$\begin{cases} Z_{\text{Maxwell}}(\tau+1) = Z_{\text{Maxwell}}(\tau) \\ Z_{\text{Maxwell}}(-\frac{1}{\tau}) = \tau^{\frac{\chi-\sigma}{4}} \bar{\tau}^{\frac{\chi+\sigma}{4}} Z_{\text{Maxwell}}(\tau). \end{cases}$$

Here,

$\chi$ : Euler number

$$\sigma = \frac{1}{3} \int \frac{1}{8\pi^2} \text{tr}(R \wedge R) : \text{signature}$$

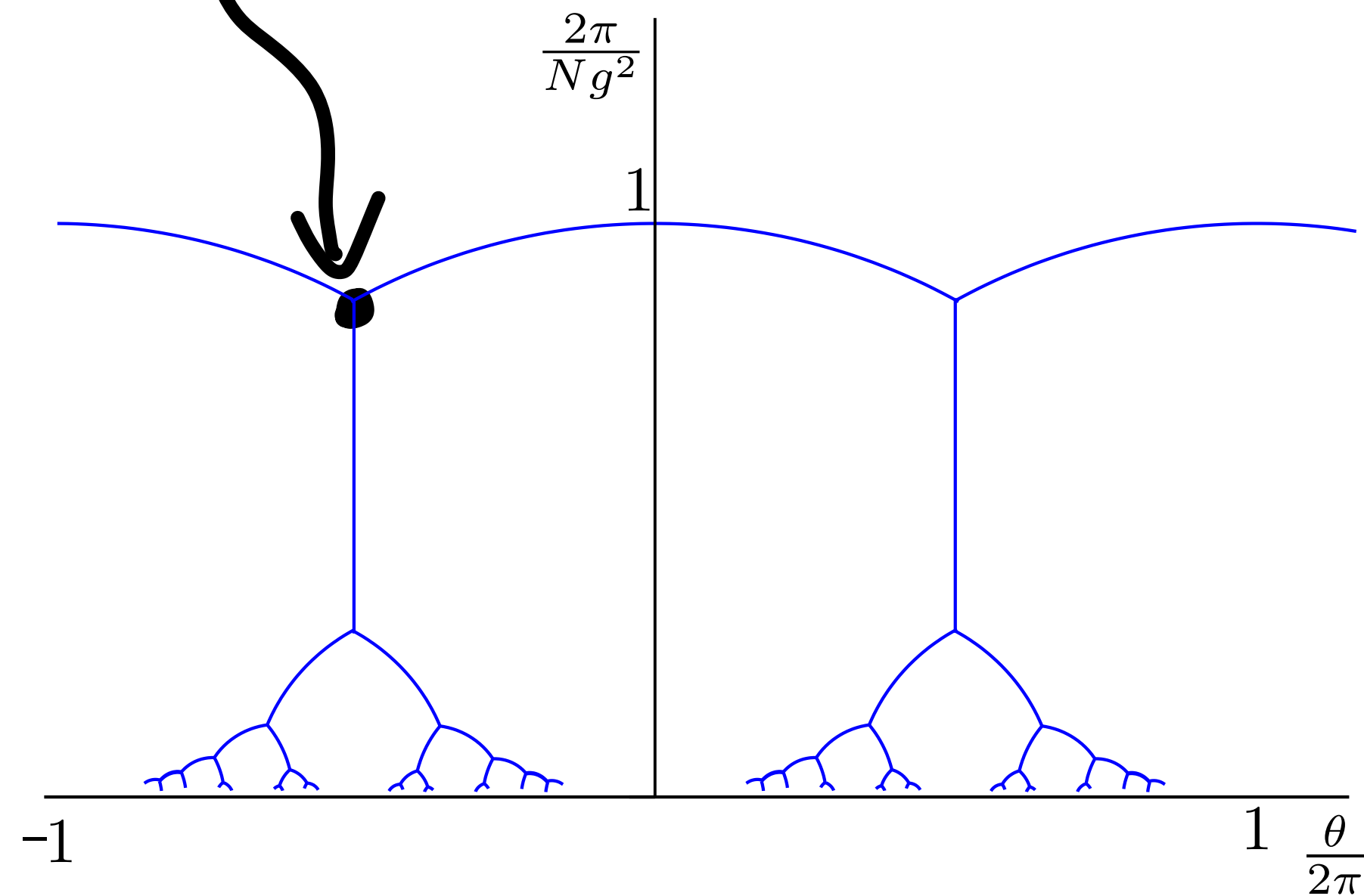


We can then show that (on K3 surface)

$$(\mathbb{Z}_6)_{ST^{-1}} : \mathbb{Z} \rightarrow e^{-\frac{2\pi i}{3}} \mathbb{Z}$$

↪ mixed gravitational anomaly.

⇒ Triple degeneracy by  $(\mathbb{Z}_6)_{ST^{-1}} \xrightarrow{SSB} \mathbb{Z}_2$  is required from anomaly.



# Summary

- Confining vacua in YM have rich top. structures.
- Cardy-Rabinovici model provides a useful playground to understand those possibilities.
- We get better knowledge on Oblique Conf.
- Self-dual CR model & its gravitational anomaly